

MULTIRESOLUTION ANALYSIS ON COMPACT RIEMANNIAN MANIFOLDS

Isaac Z. Pesenson¹

1. INTRODUCTION

The problem of representation and analysis of manifold defined functions (signals, images, and data in general) is ubiquitous in neuroscience, medical and biological applications. In the context of modeling the computations of the cortex, some twenty years ago, Mumford noted: “... *the set of higher level concepts will automatically have geometric structure*”. Indeed, in Vision input images can be thought of as points in a high-dimensional measurement space (with each input dimension corresponding to the activity of retinal ganglion cells whose axons project from the eye to the brain), however, perceptually meaningful structures lay on a manifold embedded in this space [53]. In a general context, structural and functional connectivity of the brain are usually described within network theory [Chapters XX in this volume by D. Bassett et al. [4]; by M. Pesenson [49]; and by P. Ninez et al. [36]]. However, when a network of perceptual neurons can be considered as a discrete approximation of a manifold, multiresolution analysis of manifold defined functions becomes a powerful tool.

In the last decade, the importance of these and other applications triggered the development of various generalized wavelet bases suitable for the unit spheres S^2 and S^3 and the rotation group of \mathbb{R}^3 . The goal of the present study is to describe a generalization of those approaches by constructing bandlimited and localized frames in a space $L_2(M)$, where M is a compact Riemannian manifold. The following classes of manifolds will be considered: compact manifolds without boundary, compact homogeneous manifolds, bounded domains with smooth boundaries in Euclidean spaces.

One can think of a manifold as of a surface in a Euclidean space. A homogeneous manifold is a surface with “many” symmetries like the sphere $x_1^2 + \dots + x_d^2 = 1$ in Euclidean space \mathbb{R}^d . An important example of a bounded domain is a ball $x_1^2 + \dots + x_d^2 \leq 1$ in \mathbb{R}^d .

As it will be demonstrated below, our construction of frames in a function space $L_2(M)$ heavily depends on a proper sampling of a manifold M itself. However, it is worth stressing that our main objective is the *sampling of functions on manifolds*.

In sections 2- 6 it is shown how to construct on a compact manifold (with or without boundary) a “nearly” tight bandlimited and strongly localized frame. In other words, on a very fine scale members of our frame look almost like Dirac measures. In section 7 we consider the case of compact homogeneous manifolds, i.e. which have the form $M = G/H$, where G is a compact Lie group and H is its closed subgroup. For such manifolds we are able to construct tight, bandlimited and

¹ Department of Mathematics, Temple University, Philadelphia, PA 19122; pesenson@temple.edu. The author was supported in part by the National Geospatial-Intelligence Agency University Research Initiative (NURI), grant HM1582-08-1-0019.

localized frames. The crucial role in this development is played by positive cubature formulas (Theorem 7.2) and to the product property (Theorem 7.1), which were proved in [21]. Cubature formula with positive weights in Theorem 7.2 was proved in [21] for general compact manifolds without boundary, but essentially the same proof can be used to establish it for manifolds with boundaries.

In section 8 we show that properly introduced variational splines on manifolds can be used for effective reconstruction of functions from their frame projections.

The most important fact for our construction of frames is that in a space of ω -bandlimited functions the continuous and discrete norms are equivalent. This result in the case of compact and non-compact manifolds of bounded geometry was first discovered and explored in many ways in our papers [37]-[45]. In the classical cases of straight line \mathbb{R} and circle \mathbb{S} the corresponding results are known as Plancherel-Polya and Marcinkiewicz-Zygmund inequalities. Our generalization of Plancherel-Polya and Marcinkiewicz-Zygmund inequalities implies that ω -bandlimited functions on manifolds of bounded geometry are completely determined by their values on discrete sets of points "uniformly" distributed over M with a spacing comparable to $1/\sqrt{\omega}$ and can be completely reconstructed in a stable way from their values on such sets. The last statement is obviously an extension of the famous Shannon sampling theorem to the case of Riemannian manifolds.

Our article is a summary of some results for compact manifolds that were obtained in [16]-[21], [38]-[48]. To the best of our knowledge these are the pioneering papers which contain the most general results about frames and Shannon sampling on compact and non-compact Riemannian manifolds. In particular, the paper [21] gives an end point construction of tight localized frames on homogeneous compact manifolds. The paper [47] is the first systematic development of localized frames on domains in Euclidean spaces.

Other papers which contain results about frames and wavelets on compact Riemannian manifolds are [3], [11], [34], [50]. In the setting of compact manifolds necessary conditions for sampling and interpolation in terms of Beurling-Landau densities were obtained in [35].

Applications of frames on manifolds to scattering theory, to statistics and cosmology can be found in [1], [2] [19], [20], [22], [27]-[29]. There is also a number of papers in which different kind of wavelets and frames are developed on non-compact homogeneous manifolds and in particular on Lie groups, see, e.g., [5], [6]-[9], [12]-[14], [30], [31], [37].

2. MANIFOLDS AND OPERATORS

We describe all situations which will be discussed in the paper.

2.1. Manifolds without boundary. We will work with second-order differential self-adjoint and non-negative definite operators on compact manifolds without boundary. The best known example of such operator is the Laplace-Beltrami which is given in a local coordinate system by the formula

$$Lf = \sum_{m,k} \frac{1}{\sqrt{\det(g_{ij})}} \partial_m \left(\sqrt{\det(g_{ij})} g^{mk} \partial_k f \right)$$

where g_{ij} are components of the metric tensor, $\det(g_{ij})$ is the determinant of the matrix (g_{ij}) , g^{mk} components of the matrix inverse to (g_{ij}) . Spectral properties of this operator reflect (to a certain extent) metric properties of the manifold. It

is known that Laplace-Beltrami is a self-adjoint positive definite operator in the corresponding space $L_2(M)$ constructed from g . Domains of the powers $L^{s/2}$, $s \in \mathbb{R}$, coincide with the Sobolev spaces $H^s(M)$, $s \in \mathbb{R}$. Since L is a second-order differential self-adjoint and non-negative definite operator on a compact connected Riemannian manifold (with or without boundary), then L has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2, \dots$ which goes to infinity.

2.2. Compact homogeneous manifolds. The most complete results will be obtained for compact homogeneous manifolds.

A homogeneous compact manifold M is a C^∞ -compact manifold on which a compact Lie group G acts transitively. In this case M is necessary of the form G/K , where K is a closed subgroup of G . The notation $L_2(M)$, is used for the usual Hilbert spaces, where dx is an invariant measure.

If \mathfrak{g} is the Lie algebra of a compact Lie group G then it is a direct sum $\mathfrak{g} = \mathfrak{a} + [\mathfrak{g}, \mathfrak{g}]$, where \mathfrak{a} is the center of \mathfrak{g} , and $[\mathfrak{g}, \mathfrak{g}]$ is a semi-simple algebra. Let Q be a positive-definite quadratic form on \mathfrak{g} which, on $[\mathfrak{g}, \mathfrak{g}]$, is opposite to the Killing form. Let X_1, \dots, X_d be a basis of \mathfrak{g} , which is orthonormal with respect to Q . Since the form Q is $Ad(G)$ -invariant, the operator

$$-X_1^2 - X_2^2 - \dots - X_d^2, \quad d = \dim G$$

is a bi-invariant operator on G , which is known as the Casimir operator. This implies in particular that the corresponding operator on $L_2(M)$,

$$(2.1) \quad L = -D_1^2 - D_2^2 - \dots - D_d^2, \quad D_j = D_{X_j}, \quad d = \dim G,$$

commutes with all operators $D_j = D_{X_j}$. Operator L , which is usually called the Laplace operator, is the image of the Casimir operator under differential of quazi-regular representation in $L_2(M)$. Note that if $M = G/K$ is a compact symmetric space then the number $d = \dim G$ of operators in the formula (2.1) can be strictly bigger than the dimension $m = \dim M$. For example on a two-dimensional sphere \mathbb{S}^2 the Laplace-Beltrami operator $L_{\mathbb{S}^2}$ is written as

$$(2.2) \quad L_{\mathbb{S}^2} = D_1^2 + D_2^2 + D_3^2,$$

where $D_i, i = 1, 2, 3$, generates a rotation in \mathbb{R}^3 around coordinate axis x_i :

$$(2.3) \quad D_i = x_j \partial_k - x_k \partial_j,$$

where $j, k \neq i$.

It is important to realize that in general, the operator L is not necessarily the Laplace-Beltrami operator of the natural invariant metric on M . But it coincides with such operator at least in the following cases:

1) If M is a d -dimensional torus, and $(-L)$ is the sum of squares of partial derivatives.

2) If the manifold M is itself a group G which is compact and semi-simple, then L is exactly the Laplace-Beltrami operator of an invariant metric on G ([23], Ch. II).

3) If $M = G/K$ is a compact symmetric space of rank one, then the operator L is proportional to the Laplace-Beltrami operator of an invariant metric on G/K . This follows from the fact that, in the rank one case, every second-order operator which commutes with all isometries $x \rightarrow g \cdot x$, $x \in M$, $g \in G$, is proportional to the Laplace-Beltrami operator ([23], Ch. II, Theorem 4.11).

2.3. Bounded domains with smooth boundaries. Our consideration includes also an open bounded domain $M \subset \mathbb{R}^d$ with a smooth boundary Γ which is a smooth $(d-1)$ -dimensional oriented manifold. Let $\overline{M} = M \cup \Gamma$ and $L_2(M)$ be the space of functions square-integrable with respect to the Lebesgue measure $dx = dx_1 \dots dx_d$ with the norm denoted as $\|\cdot\|$. If k is a natural number the notations $H^k(M)$ will be used for the Sobolev space of distributions on M with the norm

$$\|f\|_{H^k(M)} = \left(\|f\|^2 + \sum_{1 \leq |\alpha| \leq k} \|\partial^{|\alpha|} f\|^2 \right)^{1/2}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\partial^{|\alpha|}$ is a mixed partial derivative

$$\left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_d} \right)^{\alpha_d}.$$

Under our assumptions the space $C_0^\infty(\overline{M})$ of infinitely smooth functions with support in \overline{M} is dense in $H^k(M)$. Closure in $H^k(M)$ of the space $C_0^\infty(M)$ of smooth functions with support in M will be denoted as $H_0^k(M)$.

Since Γ can be treated as a smooth Riemannian manifold one can introduce Sobolev scale of spaces $H^s(\Gamma)$, $s \in \mathbb{R}$, as, for example, the domains of the Laplace-Beltrami operator L of a Riemannian metric on Γ .

According to the trace theorem there exists a well defined continuous surjective trace operator

$$\gamma : H^s(M) \rightarrow H^{s-1/2}(\Gamma), \quad s > 1/2,$$

such that for all functions f in $H^s(M)$ which are smooth up to the boundary the value γf is simply a restriction of f to Γ .

One considers the following operator

$$(2.4) \quad Pf = - \sum_{j,k} \partial_j (a_{j,k}(x) \partial_k f),$$

with coefficients in $C^\infty(M)$ where the matrix $(a_{j,k}(x))$ is real, symmetric and positive definite on \overline{M} . The operator L is defined as the Friedrichs extension of P , initially defined on $C_0^\infty(M)$, to the set of all functions f in $H^2(M)$ with constraint $\gamma f = 0$. The Green formula implies that this operator is self-adjoint. It is also a positive operator and the domain of its positive square root $L^{1/2}$ is the set of all functions f in $H^1(M)$ for which $\gamma f = 0$.

Thus, one has a self-adjoint positive definite operator in the Hilbert space $L_2(M)$ with a discrete spectrum $0 < \lambda_1 \leq \lambda_2, \dots$ which goes to infinity.

3. HILBERT FRAMES

If a manifold M is compact and one has an elliptic, self-adjoint positive definite operator L in $L_2(M)$ then L has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$ which goes to infinity and there exists a family $\{u_j\}$ of orthonormal eigenfunctions which form a basis in $L_2(M)$. Since eigenfunctions have perfect localization properties in the spectral domain they cannot be localized on the manifold.

It is the goal of our article to construct "better bases" in corresponding $L_2(M)$ spaces which will have rather strong localization on a manifold and in the spectral domain.

In fact, the "kind of basis" which we are going to construct is known today as a frame.

A set of vectors $\{\theta_v\}$ in a Hilbert space H is called a frame if there exist constants $A, B > 0$ such that for all $f \in H$

$$(3.1) \quad A\|f\|_2^2 \leq \sum_v |\langle f, \theta_v \rangle|^2 \leq B\|f\|_2^2.$$

The largest A and smallest B are called lower and upper frame bounds.

The set of scalars $\{\langle f, \theta_v \rangle\}$ represents a set of measurements of a signal f . To synthesize signal f from this set of measurements one has to find another (dual) frame $\{\Theta_v\}$ and then a reconstruction formula is

$$(3.2) \quad f = \sum_v \langle f, \theta_v \rangle \Theta_v.$$

Dual frame is not unique in general. Moreover it is difficult to find a dual frame.

If in particular $A = B = 1$ the frame is said to be tight or Parseval. Parseval frames are similar in many respects to orthonormal wavelet bases. For example, if in addition all vectors θ_v are unit vectors, then the frame is an orthonormal basis.

The main feature of Parseval frames is that decomposing and synthesizing a signal or image from known data are tasks carried out with the same set of functions. The important differences between frames and, say, orthonormal bases is their redundancy that helps reduce the effect of noise in data.

Frames in Hilbert spaces of functions whose members have simultaneous localization in space and frequency arise naturally in wavelet analysis on Euclidean spaces when continuous wavelet transforms are discretized. Such frames have been constructed, studied, and employed extensively in both theoretical and applied problems (in addition to papers listed in introduction one can refer to [10], [15], [26], [27]).

4. MULTIRESOLUTION AND SAMPLING

On a very general level Multiresolution Analysis on Riemannian manifolds can be described as a framework that brings together metric properties (geometry) of a manifold and spectral properties (Fourier analysis) of the corresponding Laplace-Beltrami operator.

The objective of our work is to construct a frame $\Psi_l = \{\psi_{l,j}\}$ in the space $L_2(M)$ which is somewhere between the two "extreme" bases i.e. eigenfunctions of the Laplace-Beltrami operator and a collection of Dirac functions. Multiresolution Analysis is using discretization of a manifold as a way to link geometry and analysis. To be more precise, one approximates a manifold (space) by sets of points and associates with such a discretization a frame in the space $L_2(M)$ which is localized in frequency and on manifold.

Definition 1. In the case of compact manifolds (with or without boundary) for a given $\omega > 0$, the span of eigenfunctions u_j

$$Lu_j = \lambda_j u_j$$

with $\lambda_j \leq \omega$ is denoted as $E_\omega(L)$ and is called the space of bandlimited functions on M of bandwidth ω .

According to the Weyl's asymptotic formula [24] one has

$$(4.1) \quad \dim E_\omega(L) \sim C \operatorname{Vol}(M) \omega^{d/2},$$

where $d = \dim M$ and C is an absolute constant.

As it was already mentioned, the important fact is that ω -bandlimited functions are completely determined by their values on discrete sets of points "uniformly" distributed over M with a spacing comparable to $1/\sqrt{\omega}$ and can be completely reconstructed in a stable way from their values on such sets.

Intuitively, such discrete sets can be associated with a scale $1/\sqrt{\omega}$: a finer scaling requires larger frequencies.

The main objective of Multiresolution Analysis is to construct a frame in $L_2(M)$ whose structure reflects the relation between scaling and frequency.

Now we introduce what can be considered as a notion of "points uniformly distributed over a manifold".

One can show that for a Riemannian manifold M of bounded geometry there exists a natural number N_M such that for any sufficiently small $\rho > 0$ there exists a set of points $\{y_\nu\}$ such that:

- 1) the balls $B(y_\nu, \rho/4)$ are disjoint,
- 2) the balls $B(y_\nu, \rho/2)$ form a cover of M ,
- 3) the multiplicity of the cover by balls $B(y_\nu, \rho)$ is not greater N_M .

Definition 2. A set of points $M_\rho = \{y_\nu\}$ is called a ρ -lattice if it is a set of centers of balls with the above listed properties 1)- 3).

Our main result can be described as follows.

Given a Riemannian manifold M and a sequence of positive numbers $\omega_j = 2^{2j+1}$, $j = 0, 1, \dots$, we consider the Paley-Wiener space $E_{\omega_j}(L)$ of functions bandlimited to $[0, \omega_j]$ and for a specific $c_0 = c_0(M)$ consider a set of scales

$$\rho_j = c_0 \omega_j^{-1/2}, \quad \omega_j = 2^{2j+1}, \quad j = 0, 1, \dots,$$

and construct a corresponding set of lattices

$$M_{\rho_j} = \{x_{j,k}\}_{k=1}^{m_j}, \quad x_{j,k} \in M, \quad k \in \mathbb{Z}, \quad \operatorname{dist}(x_{j,k_1}, x_{j,k_2}) \sim \rho_j,$$

of points which are distributed over M with a spacing comparable to ρ_j .

With every point $x_{j,k}$ we associate a function $\Theta_{j,k} \in L_2(M)$ such that

- (1) function $\Theta_{j,k}$, $1 \leq k \leq m_j \in \mathbb{N}$, is bandlimited to $[0, \omega_j]$;
- (2) the "essential" support of $\Theta_{j,k}$, $1 \leq j \leq m_k \in \mathbb{N}$, is in the ball $B(x_{j,k}, \rho_j)$ with center at $x_{j,k}$ and of radius ρ_j ;
- (3) the set $\bigcup_{k=1}^{m_j} \Theta_{j,k}$ is a frame in $E_{\omega_j}(L)$;
- (4) the set $\bigcup_{j=0}^{\infty} \bigcup_{k=1}^{m_j} \Theta_{j,k}$ is a frame in $L_2(M)$;

Note that $\Theta = \bigcup_{j,k} \{\Theta_{j,k}\}$ corresponds to the set $X = \bigcup_j M_{\rho_j}$ which is union of all scales.

Thus, by changing a subset of functions $\Theta_j = \bigcup_{k=1}^{m_j} \Theta_{j,k}$ to a subset $\Theta_i = \bigcup_{k=1}^{m_i} \Theta_{i,k}$ (in the space $L_2(M)$) we are actually going

- a) from the scale M_{ρ_j} to the scale M_{ρ_i} in space

and at the same time

b) from the frequency band $[0, 2^{2j+1}]$ to the frequency band $[0, 2^{2i+1}]$ in the frequency domain.

Intuitively, index $1 \leq k \leq m_j$ corresponds to Dirac measures (points on a manifold) and $j \in \mathbb{Z}$ to bands of frequencies.

5. SHANNON SAMPLING OF BANDLIMITED FUNCTIONS ON MANIFOLDS

The most important fact for our development is an analog of the Shannon's Sampling Theorem for Riemannian manifolds of bounded geometry, which was established in our papers [37]-[45]. Our generalization of the Sampling Theorem states that ω -bandlimited functions on a manifold M are completely determined by their values on sets of points distributed over M with a spacing comparable to $1/\sqrt{\omega}$ and can be completely reconstructed in a stable way from their values on such sets.

Theorem 5.1. *For a given $0 < \delta < 1$ there exists a constant $c_0 = c_0(M, \delta)$ such that, if*

$$(5.1) \quad \rho = c_0 \omega^{-1/2}, \quad \omega > 0,$$

then for any ρ -lattice $M_\rho = \{x_k\}$ there exists a set of weights $\mu_k(\rho) \sim \rho^d$ such that one has the following Plancherel-Polya inequalities (or frame inequalities)

$$(5.2) \quad (1 - \delta) \|f\|^2 \leq \sum_k \mu_k(\rho) |f(x_k)|^2 \leq \|f\|^2.$$

for all $f \in E_\omega(L)$.

The inequalities (5.2) imply that every $f \in E_\omega(L)$ is uniquely determined by its values on $M_\rho = \{x_k\}$ and can be reconstructed from these values in a stable way.

It shows that if θ_k is the orthogonal projection of the distribution $\sqrt{\mu_k(\rho)} \delta_{x_k}$ (δ_{x_k} is the Dirac measure at x_k) on the space $E_\omega(L)$, then the following frame inequalities hold

$$(1 - \delta) \|f\|^2 \leq \sum_k |\langle f, \theta_k \rangle|^2 \leq \|f\|^2$$

for all $f \in E_\omega(L)$. In other words, we obtain that the set of functions $\{\theta_k\}$ is a frame in the space $E_\omega(L)$.

According to the general theory of frames, one has that if $\{\Psi_k\}$ is a frame which is dual to $\{\theta_k\}$ in the space $E_\omega(L)$ (such frame is not unique) then the following reconstruction formula holds

$$(5.3) \quad f = \sum_k \langle f, \theta_k \rangle \Psi_k.$$

The condition (5.1) imposes a specific rate of sampling in (5.2). It is interesting to note that this rate is essentially optimal. Indeed, on one hand the Weyl's asymptotic formula (4.1) gives the dimension of the space $E_\omega(L)$. On the other hand, the condition (5.1) and the definition of a ρ -lattice imply that the number of points in an "optimal" lattice M_ρ for (5.2) can be approximately estimated as

$$\text{card } M_\rho \sim \frac{\text{Vol}(M)}{c_0^d \omega^{-d/2}} = c \text{Vol}(M) \omega^{d/2}, \quad d = \dim M,$$

which is in agreement with the Weyl's formula.

6. LOCALIZED FRAMES ON COMPACT MANIFOLDS

In this section for every $f \in L_2(M)$ we construct a special decomposition into bandlimited functions and then perform a discretization step by applying the Theorem 5.1 from the previous section.

Choose a function $\Phi \in C_c^\infty(\mathbb{R}^+)$, supported in the interval $[2^{-2}, 2^4]$ such that

$$(6.1) \quad \sum_{j=-\infty}^{\infty} |\Phi(2^{-2j}s)|^2 = 1$$

for all $s > 0$.

For example, we could choose a smooth monotonically decreasing function ψ on \mathbb{R}^+ with $0 \leq \psi \leq 1$, with $\psi \equiv 1$ in $[0, 2^{-2}]$ and with $\psi = 0$ in $[2^2, \infty)$. In this case $\psi(s/2^2) - \psi(s) \geq 0$ and we set

$$(6.2) \quad \Phi(s) = [\psi(s/2^2) - \psi(s)]^{1/2}, \quad s > 0,$$

which will have support in $[2^{-2}, 2^4]$ and will satisfy (6.1). Using the Spectral Theorem for L and the equality (6.1) one can obtain

$$(6.3) \quad \sum_{j=-\infty}^{\infty} |\Phi|^2(2^{-2j}L) = I - P,$$

where P is the projector on the kernel of L and where the sum (of operators) converges strongly on $L_2(M)$.

It should be noted, that in the case of Dirichlet boundary conditions $P = 0$.

Formula (6.3) implies the following equality

$$(6.4) \quad \sum_{j \in \mathbb{Z}} \|\Phi(2^{-2j}L)f\|_2^2 = \|(I - P)f\|_2^2.$$

Moreover, since function $\Phi(2^{-2j}s)$ has support in $[2^{2j-2}, 2^{2j+4}]$ the function $\Phi(2^{-2j}L)f$ is bandlimited to $[2^{2j-2}, 2^{2j+4}]$.

According to Theorem 5.1 for a fixed $0 < \delta < 1$ there exists a constant $c_0(M, \delta)$ such that for

$$\rho_j = c_0 \omega_j^{-1/2} = c_0 2^{-j-2}, \quad j \in \mathbb{Z},$$

and any ρ_j -lattice $M_{\rho_j} = \{x_{j,k}\}$, $1 \leq k \leq K_j$, the inequalities (5.2) hold.

Thus, if $\theta_{j,k} \in E_{\omega_j}(L) = E_{2^{2j+4}}(L)$ is the projection of $\sqrt{\mu_k(\rho)}\delta_{x_k}$ onto $E_{\omega_j}(L)$, then we have the following frame inequalities in $PW_{\omega_j}(M)$ for every $j \in \mathbb{Z}$

$$(6.5) \quad (1 - \delta) \|\Phi(2^{-2j}L)f\|^2 \leq \sum_{k=1}^{K_j} |\langle \Phi(2^{-2j}L)f, \theta_{j,k} \rangle|^2 \leq \|\Phi(2^{-2j}L)f\|^2,$$

where $\Phi(2^{-2j}L)f \in E_{\omega_j}(L) = E_{2^{2j+4}}(L)$. Together with (6.4) it gives for any $f \in L_2(M)$ the following inequalities

$$(6.6) \quad (1 - \delta) \|f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^{K_j} |\langle \Phi(2^{-2j}L)f, \theta_{j,k} \rangle|^2 \leq \|f\|^2, \quad f \in L_2(M), \quad \theta_{j,k} \in E_{2^{2j+4}}(L).$$

But, since operator $\Phi(2^{-2j}L)$ is self-adjoint,

$$\langle \Phi(2^{-2j}L)f, \theta_{j,k} \rangle = \langle f, \Phi(2^{-2j}L)\theta_{j,k} \rangle,$$

we obtain, that for the functions

$$(6.7) \quad \Theta_{j,k} = \Phi(2^{-2j}L)\theta_{j,k} \in E_{[2^{2j-2}, 2^{2j+4}]}(L) \subset E_{2^{2j+4}}(L)$$

the following frame inequalities hold

$$(6.8) \quad (1 - \delta)\|(I - P)f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{k=1}^{K_j} |\langle f, \Theta_{j,k} \rangle|^2 \leq \|(I - P)f\|^2, \quad f \in L_2(M).$$

The next goal is to find an explicit formula for the operator $\Phi(2^{-2j}L)$ and to show localization of the frame elements $\Theta_{j,k}$.

According to the Spectral Theorem if a self-adjoint positive-definite operator L has a discrete spectrum $0 < \lambda_1 \leq \lambda_2 \leq \dots$, and a corresponding set of eigenfunctions u_j ,

$$Lu_j = \lambda_j u_j,$$

which form an orthonormal basis in $L_2(M)$, then for any bounded real-valued function F of one variable one can construct a self-adjoint operator $F(L)$ in $L_2(M)$ as

$$(6.9) \quad F(L)f(x) = \int_M \mathcal{K}^F(x, y)f(y)dy, \quad f \in L_2(M),$$

where $\mathcal{K}^F(x, y)$ is a smooth function defined by the formula

$$(6.10) \quad \mathcal{K}^F(x, y) = \sum_m F(\lambda_m)u_m(x)\overline{u_m(y)}.$$

The following notations will be used

$$(6.11) \quad [F(tL)f](x) = \int_M \mathcal{K}_{\sqrt{t}}^F(x, y)f(y)dy, \quad f \in L_2(M),$$

where

$$(6.12) \quad \mathcal{K}_{\sqrt{t}}^F(x, y) = \sum_m F(t\lambda_m)u_m(x)\overline{u_m(y)}.$$

In our situation we have the formulas

$$(6.13) \quad \mathcal{K}_{2^{-j}}^\Phi(x, y) = \sum_{m \in \mathbb{Z}_+} \Phi(2^{-2j}\lambda_m)u_m(x)\overline{u_m(y)},$$

and

$$(6.14) \quad [\Phi(2^{-2j}L)f](x) = \int_M \mathcal{K}_{2^{-j}}^\Phi(x, y)f(y)dy, \quad f \in L_2(M).$$

By expanding $f \in L_2(M)$ in terms of eigenfunctions of L

$$f = \sum_{m \in \mathbb{Z}_+} c_m(f)u_m, \quad c_m(f) = \langle f, u_m \rangle,$$

one has

$$\Phi(2^{-2j}L)f = \sum_{2^{2j-2} \leq \lambda_m \leq 2^{2j+4}} \Phi(2^{-2j}\lambda_m)c_m(f)u_m.$$

After all we obtain

$$\begin{aligned} \Theta_{j,k} &= \Phi(2^{-2j}L) \theta_{j,k} = \\ (6.15) \quad \mathcal{K}_{2^{-j}}^\Phi(x, x_{j,k}) &= \sum_{m \in \mathbb{Z}_+} \Phi(2^{-2j}\lambda_m) c_m(\theta_{j,k}) \overline{u}_m(x_{j,k}) u_m(x). \end{aligned}$$

Localization properties of the kernel $\mathcal{K}_t^F(x, y)$ are given in the following statement.

Lemma 6.1. *If L is an elliptic self-adjoint second order differential operators on a compact manifold (without boundary or with a smooth boundary) and $\mathcal{K}_t^F(x, y)$ is given by (6.10), where F is an even function in $C_c^\infty(\mathbb{R})$, then on $\Omega \times \Omega \setminus \Delta$, $\Delta = \{(x, x)\}$, $x \in \Omega$, kernel $\mathcal{K}_t^F(x, y)$ vanishes to infinite order as t goes to zero.*

Different proofs can be found in [16], [21], [52], [24].

The last property shows that kernel $\mathcal{K}_t^F(x, y)$ is localized as long as F is an even Schwartz function. Indeed, if for a fixed point $x \in M$ a point $y \in M$ is "far" from x and t is small, then the value of $\mathcal{K}_t^F(x, y)$ is small.

The Lemma 6.1 is an analog of the important fact for Euclidean spaces that the Fourier transform of a Schwartz function is a Schwartz function.

Since $\mathcal{K}_t^\Phi(x, y)$ is smooth and M is bounded we can express localization of $\mathcal{K}_t^\Phi(x, y)$ by using the following inequality: for any $N > 0$ there exists a $C(N)$ such, that for all sufficiently small positive t

$$(6.16) \quad |\mathcal{K}_t^\Phi(x, y)| \leq C(N) \frac{t^{-d}}{\max(1, t^{-1}|x - y|)^N}, \quad t > 0.$$

From here one obtains the next inequality

$$\begin{aligned} |\Theta_{j,k}(x)| &= |\Phi(2^{-2j}L)\theta_{j,k}(x)| = |\mathcal{K}_{2^{-j}}^\Phi(x, x_{j,k})| \leq \\ &C(N) \frac{2^{dj}}{\max(1, 2^j|x - x_{j,k}|)^N}. \end{aligned}$$

Thus, the following statement about localization of every $\Theta_{j,k}$ holds.

Lemma 6.2. *For any $N > 0$ there exists a $C(N) > 0$ such that*

$$(6.17) \quad |\Theta_{j,k}(x)| \leq C(N) \frac{2^{dj}}{\max(1, 2^j|x - x_{j,k}|)^N},$$

for all $j \in \mathbb{Z}$.

Inequality (6.8) and Lemma 6.2 give the following Frame Theorem.

Theorem 6.3. *For a given $0 < \delta < 1$ there exists a constant $c_0 = c_0(M, \delta)$ such that, if*

$$\rho_j = c_0 2^{-j-2}, \quad \omega > 0, \quad j \in \mathbb{Z},$$

and $M_{\rho_j} = \{x_{j,k}\}$, is a ρ_j -lattice, then the corresponding set of functions $\{\Theta_{j,k}\}$:

$$\Theta_{j,k} = \Phi(2^{-2j}L) \theta_{j,k}, \quad j \in \mathbb{Z}, \quad 1 \leq k \leq K_j,$$

where $\theta_{j,k}$ is projection of the measure $\sqrt{\mu_{j,k}(\rho_j)} \delta_{x_{j,k}}$ onto $E_{2^{2j+4}}(L)$, is a frame in $L_2(M)$ with constants $1 - \delta$ and 1.

In other words, the following frame inequalities hold

$$(1 - \delta)\|f\|^2 \leq \sum_{j \in \mathbb{Z}} \sum_{1 \leq k \leq K_j} |\langle f, \Theta_{j,k} \rangle|^2 \leq \|f\|^2,$$

for all $f \in L_2(M)$.

Every $\Theta_{j,k}$ is bandlimited to $[2^{2j-2}, 2^{2j+4}]$ and in particular belongs to $E_{2^{2j+4}}(L)$. Localization properties of $\Theta_{j,k}$ are described in Lemma 6.2.

7. PARSEVAL FRAMES ON HOMOGENEOUS MANIFOLDS

In this section we assume that a manifold M is homogeneous (has many symmetries) in the sense that it is of the form $M = G/H$, where G is a compact Lie group and H is its closed subgroup. In this situation we construct spaces of bandlimited functions by using the Casimir operator L that was defined in (2.1).

Under these assumptions we are able to construct a tight bandlimited and localized frame in the space $L_2(M)$.

Theorem 7.1. (Product property [21]) *If $M = G/H$ is a compact homogeneous manifold and L is the same as above, then for any f and g belonging to $E_\omega(L)$, their product fg belongs to $E_{4m\omega}(L)$, where m is the dimension of the group G .*

Theorem 7.2. (Cubature formula [21]) *There exists a positive constant a_0 , such that if $\rho = a_0(\omega + 1)^{-1/2}$, then for any ρ -lattice M_ρ , there exist strictly positive coefficients $\alpha_{x_k} > 0$, $x_k \in M_\rho$, for which the following equality holds for all functions in $E_\omega(M)$:*

$$(7.1) \quad \int_M f dx = \sum_{x_k \in M_\rho} \alpha_{x_k} f(x_k).$$

Moreover, there exists constants c_1, c_2 , such that the following inequalities hold:

$$(7.2) \quad c_1 \rho^d \leq \alpha_{x_k} \leq c_2 \rho^d, \quad d = \dim M.$$

Using the same notations as in the previous section we find

$$(7.3) \quad \sum_{j=-\infty}^{\infty} \|\Phi(2^{-2j}L)f\|_2^2 = \|(I - P)f\|_2^2$$

Expanding $f \in L_2(M)$ in terms of eigenfunctions of L

$$f = \sum_i c_i(f) u_i, \quad c_i(f) = \langle f, u_i \rangle,$$

we have

$$\Phi(2^{-2j}L)f = \sum_i \Phi(2^{-2j}\lambda_i) c_i(f) u_i.$$

Since for every j function $\Phi(2^{-2j}s)$ is supported in the interval $[2^{2j+2}, 2^{2j+4}]$ the function $\Phi(2^{-2j}L)f$ is bandlimited and belongs to $E_{2^{2j+4}}(L)$.

But then the function $\overline{\Phi(2^{-2j}L)f}$ is also in $E_{2^{2j+4}}(L)$. Since

$$|\Phi(2^{-2j}L)f|^2 = [\Phi(2^{-2j}L)f] \left[\overline{\Phi(2^{-2j}L)f} \right],$$

one can use the product property to conclude that

$$|\Phi(2^{-2j}L)f|^2 \in E_{4m2^{2j+4}}(L),$$

where $m = \dim G$, $M = G/H$.

To summarize, we proved, that for every $f \in L_2(M)$ we have the following decomposition

$$(7.4) \quad \sum_{j=-\infty}^{\infty} \|\Phi(2^{-2j}L)f\|_2^2 = \|(I-P)f\|_2^2, \quad |\Phi(2^{-2j}L)f|^2 \in E_{4m2^{2j+4}}(L).$$

The next objective is to perform a discretization step. According to our result about cubature formula there exists a constant $a_0 > 0$ such that for all integer j if

$$(7.5) \quad \rho_j = a_0(4m2^{2j+4} + 1)^{-1/2} \sim 2^{-j}, \quad m = \dim G, \quad M = G/H,$$

then for any ρ_j -lattice M_{ρ_j} one can find coefficients $b_{j,k}$ with

$$(7.6) \quad b_{j,k} \sim \rho_j^d, \quad d = \dim M,$$

for which the following exact cubature formula holds

$$(7.7) \quad \|\Phi(2^{-2j}L)f\|_2^2 = \sum_{k=1}^{J_j} b_{j,k} |\Phi(2^{-2j}L)f(x_{j,k})|^2,$$

where $x_{j,k} \in M_{\rho_j}$, ($k = 1, \dots, J_j = \text{card}(M_{\rho_j})$).

Now, for $t > 0$, let \mathcal{K}_t^Φ be the kernel of $\Phi(t^2L)$, so that, for $f \in L_2(M)$,

$$(7.8) \quad [\Phi(t^2L)]f(x) = \int_M \mathcal{K}_t^\Phi(x, y) f(y) dy.$$

For $x, y \in M$, we have

$$(7.9) \quad \mathcal{K}_t^\Phi(x, y) = \sum_i \Phi(t^2\lambda_i) u_i(x) \overline{u_i(y)}.$$

Corresponding to each $x_{j,k}$ we now define the functions

$$(7.10) \quad \psi_{j,k}(y) = \overline{\mathcal{K}_{2^{-j}}^\Phi(x_{j,k}, y)} = \sum_i \overline{\Phi(2^{-2j}\lambda_i)} \overline{u_i(x_{j,k})} u_i(y),$$

$$(7.11) \quad \Psi_{j,k} = \sqrt{b_{j,k}} \psi_{j,k}.$$

We find that for all $f \in L_2(M)$,

$$(7.12) \quad \|(I-P)f\|_2^2 = \sum_{j,k} |\langle f, \Psi_{j,k} \rangle|^2.$$

Note that, by (7.10), (7.11), and the fact that $\Phi(0) = 0$, each $\Psi_{j,k} \in (I-P)L_2(M)$.

Thus the following statement is proved.

Theorem 7.3. *If M is a homogeneous manifold, then the set of functions $\{\Psi_{j,k}\}$, constructed in (7.11) is a Parseval frame in the space $(I-P)L_2(M)$.*

Here, functions $\Psi_{j,k}$ belong to $E_{\omega_j}(L)$ and their spatial decay follows from Lemma 6.1.

By general frame theory, if $f \in L_2(M)$, we have

$$(7.13) \quad (I-P)f = \sum_{j=\Omega}^{\infty} \sum_k \langle f, \Psi_{j,k} \rangle \Psi_{j,k} = \sum_{j=\Omega}^{\infty} \sum_k b_{j,k} \langle f, \psi_{j,k} \rangle \psi_{j,k},$$

with convergence in $L_2(M)$.

8. VARIATIONAL SPLINES ON MANIFOLDS

As it was explained (see the formula (7.13)) one can always use a dual frame for reconstruction of a function from its projections. However, in general it is not easy to construct a dual frame (unless the frame is tight and then it is self-dual).

The goal of this section is to introduce variational splines on manifolds and to show that such splines can be used for reconstruction of bandlimited functions from appropriate sets of samples.

Given a ρ lattice $M_\rho = \{x_\gamma\}$ and a sequence $\{z_\gamma\} \in l_2$ we will be interested in finding a function s_k in the Sobolev space $H^{2k}(M)$, where $k > d/2$, $d = \dim M$, such that

- (1) $s_k(x_\gamma) = z_\gamma, x_\gamma \in M_\rho$;
- (2) function s_k minimizes functional $g \rightarrow \|L^k g\|_{L_2(M)}$.

For a given sequence $\{z_\gamma\} \in l_2$ consider a function f from $H^{2k}(M)$ such that $f(x_\gamma) = z_\gamma$. Let $\mathcal{P}f$ denote the orthogonal projection of this function f in the Hilbert space $H^{2k}(M)$ with the inner product

$$\langle f, g \rangle = \sum_{x_\gamma \in M_\rho} f(x_\gamma)g(x_\gamma) + \langle L^{k/2} f, L^{k/2} g \rangle$$

on the subspace $U^{2k}(M_\rho) = \{f \in H^{2k}(M) | f(x_\gamma) = 0\}$ with the norm generated by the same inner product. Then the function $g = f - \mathcal{P}f$ will be the unique solution of the above minimization problem for the functional $g \rightarrow \|L^k g\|_{L_2(M)}, k = 2^l d$.

It is convenient to introduce the so-called Lagrangian splines. For a point x_γ in a lattice M_ρ the corresponding Lagrangian spline l_γ^{2k} is a function in $H^{2k}(M)$ that minimizes the same functional and takes value 1 at the point x_γ and 0 at all other points of M_ρ . Different parts of the following theorem can be found in [38], [39], [41].

Theorem 8.1. *The following statements hold:*

- (1) *for any function f from $H^{2k}(M)$, $k = 2^l d$, $l = 1, 2, \dots$, there exists a unique function $s_k(f)$ from the Sobolev space $H^{2k}(M)$, such that $f|_{M_\rho} = s_k(f)|_{M_\rho}$; and this function $s_k(f)$ minimizes the functional $u \rightarrow \|L^k u\|_{L_2(M)}$;*
- (2) *every such function $s_k(f)$ is of the form*

$$s_k(f) = \sum_{x_\gamma \in M_\rho} f(x_\gamma) l_\gamma^{2k};$$

- (3) *functions l_γ^{2k} form a Riesz basis in the space of all polyharmonic functions with singularities on M_ρ i.e. in the space of such functions from $H^{2k}(M)$ which in the sense of distributions satisfy equation*

$$L^{2k} u = \sum_{x_\gamma \in M_\rho} \alpha_\gamma \delta(x_\gamma)$$

where $\delta(x_\gamma)$ is the Dirac measure at the point x_γ ;

- (4) *if in addition the set M_ρ is invariant under some subgroup of diffeomorphisms acting on M then every two functions $l_\gamma^{2k}, l_\mu^{2k}$ are translates of each other.*

Next, if $f \in H^{2k}(M)$, $k = 2^l d$, $l = 0, 1, \dots$ then $f - s_k(f) \in U^{2k}(M_\rho)$ and we have for $k = 2^l d$, $l = 0, 1, \dots$

$$\|f - s_k(f)\|_{L_2(M)} \leq (C_0 \rho)^k \|L^{k/2}(f - s_k(f))\|_{L_2(M)}.$$

Using minimization property of $s_k(f)$ we obtain the inequality

$$(8.1) \quad \left\| f - \sum_{x_\gamma \in M_\rho} f(x_\gamma) l_{x_\gamma} \right\|_{L_2(M)} \leq (c_0 \rho)^k \|L^{k/2} f\|_{L_2(M)}, \quad k = 2^l d, \quad l = 0, 1, \dots,$$

and for $f \in E_\omega(L)$ the Bernstein inequality gives for any $f \in E_\omega(L)$ and $k = 2^l d$, $l = 0, 1, \dots$,

$$(8.2) \quad \left\| f - \sum_{x_\gamma \in M_\rho} f(x_\gamma) l_{x_\gamma} \right\|_{L_2(M)} \leq (c_0 \rho \sqrt{\omega})^k \|f\|_{L_2(M)}.$$

These inequalities lead to the following Approximation and Reconstruction Theorem.

Theorem 8.2. *There exist constants $C = C(M) > 0$ and $c_0 = c_0(M) > 0$ such that for any $\omega > 0$ and any M_ρ with $0 < \rho \leq c_0 \omega^{-1/2}$ the following inequality holds for all $f \in E_\omega(L)$*

$$\sup_{x \in M} |(s_k(f)(x) - f(x))| \leq \omega^d (C(M) \rho^2 \omega)^{k-d} \|f\|, \quad k = (2^l + 1)d, \quad l = 0, 1, \dots$$

In other words, by choosing $\rho > 0$ such that

$$\rho < (C(M) \omega)^{-1/2},$$

one obtains the following reconstruction algorithm

$$f(x) = \lim_{k \rightarrow \infty} s_k(f)(x),$$

where convergence holds in the uniform norm.

It should be noted that there exists an algorithm [41] which allows to express variational splines in terms of eigenfunctions of the operator L . Moreover, it was also shown [39] that eigenfunctions of L that belong to a fixed space $E_\omega(L)$ can be perfectly approximated by eigenfunctions of certain finite-dimensional matrices in spaces of splines with a fixed set of nodes.

9. CONCLUSION

The analysis of functions defined on manifolds is of central importance not only to neuroscience (studies of vision, speech, and motor control), but also to population genetics, finding patterns in gene data [25], [32], and manifold models for general signals and images [51]. The present study expands the well-developed field of time-frequency analysis based on wavelets, frames and splines from Euclidean spaces to compact Riemannian manifolds thus giving the means for modeling various important complex phenomena.

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